Descriptive Set Theory
Lecture 8

In particular, every 0-dim. 2nd ctbl space admits a ctbl basis of clopen sets.

Topological characterizations of our champion 0-dim Polish spaces

Char. of $2^{\mathbb{N}}$ (Brouwer). The Cantor space $2^{\mathbb{N}}$ is the unique 0-dim perfect compact metrizable nonempty top space (up to homeomorphism).

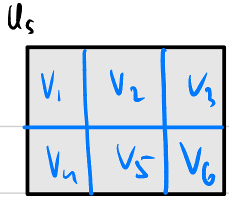
Proof. Fix such a space X and build a homeom $2^{\mathbb{N}} \xrightarrow{\sim} X$. We construct a Cantor scheme $(U_s)_{s \in 2^{<\mathbb{N}}}$ of vanishing diam. s.t.

(i) $U_s = U_{s0} \cup U_{s1}$

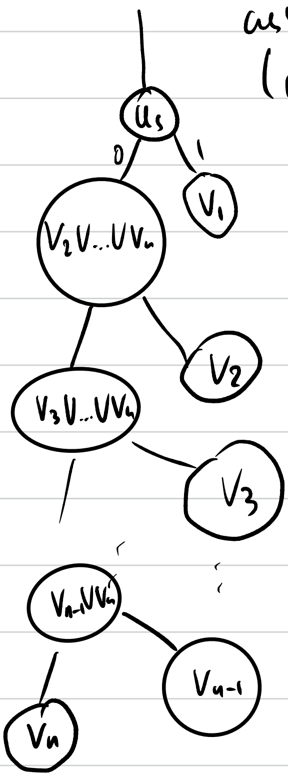
(ii) $U_s \neq \emptyset$ clopen.

Given this, because $\overline{U_{si}} = U_{si} \subseteq U_s$ and $U_s \neq \emptyset$ assures that the domain of the induced map is $2^{\mathbb{N}}$, condition (i) ensures surjectivity, and f is open because U_s is open (also, automatically, because $2^{\mathbb{N}}$ is compact).

We now build such a scheme inductively. Let $U_\emptyset := X$.



Suppose U_s is defined and satisfies (ii). Cover U_s with clopen subsets of diameter $\le 2^{-|s|}$ and by compactness, there is a finite subcover: $V_1, V_2, \dots, V_n$. WLOG, assume these are nonempty. Also, because $|U_s| \geq 2$ by perfectness, we may assume WLOG that $n \geq 2$. Finally, because the V_i are clopen, assume WLOG that they are pairwise disjoint (replace V_i with $V_i \setminus (V_1 \cup V_2 \cup \dots \cup V_{i-1})$). Continue, considering the next shortest $t \in 2^{<\mathbb{N}}$ s.t. U_t is defined but U_{t0} and U_{t1} aren't. $\square$



(Alexandrov-Urysohn)

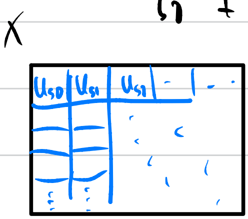
Char. of $\mathbb{N}^{\mathbb{N}}$ ✓. Baire space $\mathbb{N}^{\mathbb{N}}$ is the unique 0-dim perfect far-from-compact Polish space (up to homeomorphism), where far-from-compact means that every compact subset has empty interior.

In particular, with no work we get that $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

0-dim Polish as closed subsets of $\mathbb{N}^{\mathbb{N}}$.

Theorem. Every 0-dim Polish space X is homeomorphic to a closed subset of $\mathbb{N}^{\mathbb{N}}$. In particular, it's of the form $[T]$ for some tree T on $\mathbb{N}$.

Proof. We build a Luzin scheme $(U_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of vanishing diameter s.t. (i) $U_s = \bigcup_{i \in \mathbb{N}} U_{si}$.
(ii) U_s is clopen.
This ensures that the induced map f is open and, hence $\overline{U_{si}} = U_{si} \subseteq U_s$ and the diameter vanishes, the domain of f is closed. (i) ensures surjectivity, so f is a homeo of a closed subset of $\mathbb{N}^{\mathbb{N}}$ with X.



Let $U_\emptyset := X$. Suppose U_s is defined and write it as a disjoint union of clopen sets $U_{s0}, U_{s1}, U_{s2}, \dots$ of diam $< 2^{-|s|}$.
(Some of these U_{si}-s may be empty.) $\square$

Parametrizing every Polish space with $\mathbb{N}^{\mathbb{N}}$.

Theorem. Every nonempty Polish space is a continuous image of $\mathbb{N}^\mathbb{N}$. In fact, every Polish space is a continuous injective image of a closed subset of $\mathbb{N}^\mathbb{N}$ (but not necessarily via an open map).

Proof. The "in fact" statement implies the main statement because every nonempty closed subset of $\mathbb{N}^\mathbb{N}$ is a retract of $\mathbb{N}^\mathbb{N}$. Now let X be a Polish space. We build a Luzin scheme $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ of vanishing diam s.t.

(i) $F_s = \bigsqcup_{i\in\mathbb{N}} F_{si}$ $\left(= \bigcup_{i\in\mathbb{N}} \overline{F_{si}}\right.$, hence automatically F_σ).

(ii) $\overline{F_{si}} \subseteq F_s$. (iii) F_s is F_σ.

Granted this, the induced map would be a surjective continuous injection with closed domain.

False attempt

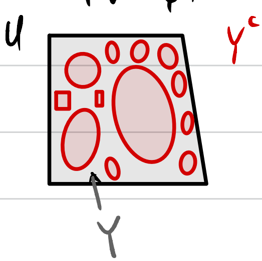
Take $F_\emptyset := X$. We write X as a union of open sets of small diameter and then disjointify, getting some Boolean combination of open sets: $F_0, F_1, \dots$

In the next step, take F_0 and again we can write it a a disjoint union of small diam. sets: $F_{00}, F_{01}, \dots$ but how would we ensure that $\overline{F_{0i}} \subseteq F_0$? This is hard because F_0 is neither open nor closed.

After adding (iii), we suppose that F_s is defined and is F_σ, i.e. $F_s = \bigcup_{n\in\mathbb{N}} C_n$. We write each C_n as a union of Boolean combinations of open sets $C_{n0}, C_{n1}, C_{n2}, \dots$ of diam $< 2^{-|s|}$ so $F_s = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} C_{ni}$. Now we disjointify all these sets, and obtain $F_s = \bigsqcup_{m\in\mathbb{N}} D_m$ where each D_m is still a Boolean combo of open, hence F_σ. Put $F_{sm} := D_m$. (Here we used $\mathbb{N}\times\mathbb{N} \cong \mathbb{N}$.) $\square$

Baire category

Nowhere dense sets. Let X be a top. space. A set $Y \subseteq X$ is is *somewhere dense* if $\exists$ nonempty open $U \subseteq X$ s.t. $Y \cap U$ is dense in U. Thus, we call a set $Y \subseteq X$ *nowhere dense* if it's not somewhere dense, i.e. $\forall\, \emptyset \neq$ open $U \subseteq X$, $Y \cap U$ is not dense in U, i.e. $\exists\, \emptyset \neq$ open $V \subseteq U$ s.t. $Y \cap V = \emptyset$.



Prop. For a top. space X and $Y \subseteq X$, TFAE:

(1) Y is nowhere dense.

(2) For every $\emptyset \neq$ open set $U \subseteq X$, $\exists\, \emptyset \neq$ open $V \subseteq U$

s.t. $V \cap Y = \emptyset$.

(3) $\bar{Y}$ has empty interior.

Proof. (3) $\Rightarrow$ (1). Note that $\mathrm{int}(\bar{Y}) = \emptyset \Rightarrow \bar{Y}$ is nowhere dense hence Y is nowhere dense.

(1) $\Rightarrow$ (3). If Y is not dense in any $\emptyset \neq$ open set that the closure won't contain a $\emptyset \neq$ open set. □

Prop. Let X be a top. space and $Y, U \subseteq X$.

(a) A is nowhere dense $\Leftrightarrow$ $\bar{A}$ is nowhere dense.

(b) If U is open, then $\partial U := \bar{U} \setminus U$ is nowhere dense.

(c) If U is dense open, then U^c is closed nowhere dense.

Proof. $U^c = \partial U$, so by (b). □

(d) Nowhere dense sets form an ideal, i.e. are closed under finite unions.

Proof. U A^c, B^c for any A, B nowhere dense. □